Variable Structure Control of Systems With Nonlinear Friction

H. G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University
Philadelphia, PA 19104
hkwatny@coe.drexel.edu

C. Teolis
Techno-Sciences, Incorporated
10001 Derekwood Lane, Suite 204
Lanham, MD 20706
carole@technosci.com

M. Mattice
Advanced Drives and Weapon Stabilization Lab
U. S. Army ARDEC
Picatinny Arsenal, NJ 07806
mmattice@pica.army.mil

Abstract: A new approach to control system design for systems containing uncertain, non-smooth friction is proposed. The method is based on a multi-state backstepping approach to variable structure control design. Stability and robustness properties are investigated and an example is given.

Keywords: nonlinear robust control, variable structure control, backstepping, nonlinear friction, electro-mechanical systems.

1 Introduction

Many important systems contain so-called ‘hard’ or ‘nonsmooth’ nonlinearities such as dead zone, backlash, hysteresis and coulomb friction. These nonlinearities can have a profound influence on the performance of a control system. While there exist standard models for these frequently neglected (often considered parasitic) effects, the parameters associated with them are almost always highly uncertain. Our specific interest has been applications to various pointing control systems associated with relatively small (Apache helicopter) to very large (Abrams tank) weapons. In these cases friction is a very significant issue and, depending on the drive system, backlash may also be important. In this paper we focus on the problem of nonlinear friction.

Approaches to control system design that directly address hard nonlinearities must account for the inevitable uncertainty. Several robust control alternatives have been suggested including a variety of adaptive [1, 2] and variable structure control methods. Both adaptive and variable structure control designs are simple and effective if the system is input-output feedback linearizable and minimum phase [3-7]. When this is the case, the first step in design is usually the reduction of the system a regular form. The basic reduction process applies to affine systems that are sufficiently smooth so that functions can be differentiated an appropriate number of times. In the present case we are interested in a more general class of models in which the system dynamics, particularly the uncertain components, are not smooth. Specifically, we will consider single-input single-output systems (SISO) of the form:

$$\begin{aligned} \dot{x} &= f(x) + \delta f(x,t) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where the uncertainty $\delta f(x,t)$ is piecewise continuous and the nominal system (f, g, h) is smooth and input output linearizable. Uncertainty in g and multivariable systems can be addressed by the methods of this paper but for clarity of exposition we have limited the present discussion.

In Section 2 we address variable structure (VS) control of systems with matched uncertainty. VS systems are known to be robust with respect to matched uncertainty. However, we craft an important result for 'smoothed' VS controllers that can be applied to nonmatched uncertainties via a backstepping method. The backstepping method is formulated in Section 4 after a motivating example is discussed in Section 3. Section 5 contains some concluding remarks.

2 VS Control with Matched Uncertainty

There are 2 basic steps to designing a variable structure control. The first is the design of the sliding control or equivalently the sliding surface. The second is the design of the reaching or switching control. The system is typically reduced to normal, or regular, form before the design begins. In this section, these methods are summarized.

Normal Form

Denote the $k^{\underline{\text{th}}}$ Lie (directional) derivative of the scalar function $\phi(x)$ with respect to the vector field $f(x)$ by $L_f^k(\phi)$. Now, by successive differentiation of the outputs y in (1), (assuming, for the moment, differentiability) we arrive at the following definitions for the relative degree r, and the functions $\alpha(x)$ and $\rho(x)$:

$$r := \inf\{k \mid L_g\ (L_f^{k-1}(h)) \neq 0\}$$

$$\alpha(x) := L_f^r(h\ )$$

$$\rho\ (x) := L_g\ (L_f^{r\ -1}(h\ ))$$

Also define the partial state transform $x \to z \in R^r$ as

$$z := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} \tag{2a}$$

where

$$z_k(x) = L_f^{k-1}(h\ ),\ k = 1,..,r \tag{2b}$$

It is a straightforward calculation to verify that the variables z defined by (2) satisfy the relation

$$\dot{z} = Az + b[\alpha(x) + \rho(x)u]$$

$$y = cz$$

where $$A = \begin{bmatrix} 0 & I_{r\ -1} \\ 0 & 0 \end{bmatrix},\ \ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ \ c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

The variables *z* are referred to as the linearizable coordinates. The remaining part of the transform can be defined by arbitrarily choosing additional independent coordinates. The condition $\rho(x) \neq 0$ insures the existence of a local (around x_0) change of coordinates $x \to (\xi, z)$, $\xi \in R^{n-r}, z \in R^r$ such that

$$\dot{\xi} = F(\xi, z) \tag{3a}$$

$$\dot{z} = Az + b[\alpha(x(\xi, z)) + \rho(x(\xi, z))u] \tag{3b}$$

$$y = cz \tag{3c}$$

Equation (3) is frequently referred to as the *local normal form* of (1). It is common to refer to (3a) as the *internal dynamics* and (3b) as the *linearizable dynamics*. If z is set to zero in (3a) then we have a local representation of the zero dynamics.

Equation (3) is the point of departure for the variable structure design as described in [6]. It constitutes a *regular form* in the sense of [8]. In order to accommodate nondifferentiable systems we will take as our starting point the system:

$$\dot{z} = Az + b[\alpha(x(\xi,z)) + \Delta(\xi,z,t) + \rho(x(\xi,z))u] \tag{4}$$

where Δ is a bounded function that can represent uncertainties, disturbances and/or nondifferentiable functions. We assume

$$\left|\Delta(x(\xi,z),t)\right| < \sigma_\Delta(\xi,z), \quad \forall t$$

where $\sigma_\Delta > 0$ is a continuous function.

VS Control Design

The reduction to this normal form is commonly associated as the first step in the process of feedback linearization. Here instead of feedback linearization, we construct a variable structure control law with switching surface of the form, $s(x)=Kz(x)$, where K is chosen to stabilize the sliding mode dynamics. We can prove that during sliding, the equivalent control is $u_{eq} = Kz$, so that we achieve feedback linearized behavior in the sliding phase (see, [6, 7, 9, 10]).

The second step in VS control system design is the specification of the control functions $u^{\pm}(x)$ such that the manifold s(x)=0 contains a stable submanifold which insures that sliding occurs. Now, we assume that both α (as well as Δ) is bounded by a continuous function, $|\alpha(x)| < \sigma_\alpha(x)$. There are many ways of approaching the reaching design problem, Utkin [11]. We proceed as follows. Consider the positive definite quadratic form in s

$$V(x) = s^T Q s$$

A sliding mode exists on a submanifold of $s(x)=0$ which lies in a region of the state space on which the time rate of change V is negative. Upon differentiation we obtain

$$\frac{d}{dt}V = 2\dot{s}^T Qs = 2\left[KAz + \alpha + \Delta\right]^T QKz + 2u^T \rho^T QKx$$

Now, choose

$$u = -\sigma(x)\,\mathrm{sgn}(s^*(x))\,, \quad \sigma(x) > \overline{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x)$$

$$s^*(x) = \rho^T(x)QKz(x)$$

so that

$$\dot{V} \le \overline{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x) - \sigma(x)\left|\mathrm{sgn}(s^*(x)/\varepsilon)\right| \qquad (5)$$

In this case it follows that $\dot{V}$ is negative, so that the sliding manifold is indeed attractive.

Smooth Approximation of VS Controllers

Suppose now that the switch is replaced by a smooth version of a switch. Specifically, $\mathrm{sgn}(s) \to \tanh(s/\varepsilon)$, $\varepsilon > 0$ so that

$$u = -\sigma(x)\,\mathrm{sgn}(s^*(x)) \to -\sigma(x)\tanh(s^*(x)/\varepsilon)$$

Then $\dot{V}$ is not necessarily negative for *s* small. However, for any given $\delta > 0$ there exists a sufficiently small $\varepsilon > 0$ such that $\dot{V} < 0$, for $|s| > \delta$ all trajectories enter the strip $|s(x)| < \delta$. We wish to establish more than that. Namely, we will show that the smoothed control steers the state into a neighborhood of $z = 0$ the size of which shrinks with the design (smoothing) parameter ε.

Proposition 1: Consider the system

$$\dot{z} = Az + b[\alpha(x(\xi,z)) + \Delta(x(\xi,z),t) + \rho(x(\xi,z))u]$$

Suppose that

1) bound on α, $|\alpha(x)| < \sigma_\alpha(x)$

2) bound on Δ, $|\Delta(x,t)| < \sigma_\Delta(x), \quad \forall t$

3) $K = \begin{bmatrix} a_1 & a_2 & \dots & a_{r-1} & 1 \end{bmatrix}$, where the coefficients are chosen such that the matrix

$$A_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_{r-1} \end{bmatrix} \text{ is stable}$$

4) $u = -\sigma(x)\tanh(s^*(x)/\varepsilon)$, where $\sigma(x) > \overline{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x)$ and $s^*(x) = \rho^T(x)QKz(x)$

Then for any $\delta > 0$ there exists a sufficiently small $\varepsilon > 0$ such that all trajectories enter the ball $\|z\| < \delta$ in finite time.

Proof:

Since $Kb = 1$, we can divide the state space into $\mathrm{Im}\, b \oplus \ker K$. Thus, we define a transformation:

$$z = \begin{bmatrix} b & K_{\ker} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

where the columns of $K_{\ker}$ span $\ker K$. Notice that we can choose Ω such that

$$\begin{bmatrix} K \\ \Omega \end{bmatrix} \begin{bmatrix} b & K_{\ker} \end{bmatrix} = I, \; K[I - bK] = 0, \; \Omega[I - bK] = \Omega$$

In these new coordinates the evolution equations are

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} KAb & KAK_{\ker} \\ \Omega Ab & \Omega AK_{\ker} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\alpha(x) + \Delta(x,t) - \rho(x)\sigma(x)\tanh(s^*(x)/\varepsilon) \right)$$

In addition, $s = Kz = \zeta_1$. Furthermore, $\mathrm{Re}\,\lambda(\Omega AK_{\ker}) < 0$ by design ($\Omega AK_{\ker} \sim A_s$).

Hence, there exists matrices, $Q_0 \geq 0, R \geq 0$ such that

1. $z^T Q_0 z = 0$ for $z \in \mathrm{Im}\, b$ and $z^T Q_0 z > 0$ otherwise.
2. $d(z^T Q_0 z)/dt = -z^T Rz \leq -\lambda_{\min} \|\zeta_2\|^2$, where $\lambda_{\min}$ is the smallest nonzero eigenvalue of R.

Now, consider the Liapunov function

$V(z) = z^T Q_0 z + (Kz)^T QKz > 0$ for $\|z\| \neq 0$.

$$\frac{d}{dt}V = 2\dot{z}Q_0 z + 2\dot{s}^T Qs$$
$$= 2\left\{Az + b[\alpha + \Delta + \rho u]\right\}^T Q_0 z + 2\left[KAz + \alpha + \Delta\right]^T QKz + 2u^T \rho^T QKx$$
$$\frac{d}{dt}V = 2\left\{Az\right\}^T Q_0 z + 2\left[KAz + \alpha + \Delta\right]^T QKz + 2u^T \rho^T QKx$$

Now, we have

$$2\left[KAz + \alpha + \Delta\right]^T QKz + 2u^T \rho^T QKx \le \overline{\sigma}(KA)\left\|z(x)\right\| + \sigma_\alpha(x) + \sigma_\Delta(x) - \sigma(x)\left|\tanh(s^*(x)/\varepsilon)\right|$$

and

$$2\left\{Az\right\}^T Q_0 z \le -\lambda_{\min}\left\|\zeta_2\right\|^2$$

so that

$$\frac{d}{dt}V \le -\lambda_{\min}\left\|\zeta_2\right\|^2 + \left[\hat{\sigma} - \sigma\left|\tanh(s^*/\varepsilon)\right|\right]$$

Thus, since $\sigma > \hat{\sigma}$, for any specified δ there is an ε such that $\dot{V} \le -c < 0$. Consequently, we have all trajectories entering the strip $|s| < \delta(\varepsilon)$ in finite time. In fact, for any given $\delta > 0$ there exists a corresponding sufficiently small $\varepsilon > 0$.

Now, since $s = \zeta_1$, it follows that $|s| < \delta \Rightarrow |\zeta_1| < \delta$. Consequently, from the evolution equations and since $\Omega AK_{\ker}$ is asymptotically stable we can conclude that all trajectories enter a ball with radius proportional to δ in finite time. ■

3 Controller Design with Nonsmooth Plants

One approach to dealing with nonsmooth nonlinearities is to approximate the nonsmooth function by a smooth one. In particular, we might consider replacing a piecewise smooth function $f(x)$ by a smooth ε-approximation $\hat{f}(x,\varepsilon)$ such that $\lim_{\varepsilon\to 0}\hat{f}(x,\varepsilon) \to f(x)$. Then the design proceeds using the approximate system with ε sufficiently small. It is important to realize that there is no *a priori* assurance that the resulting control system when applied to the original nonsmooth plant will produce closed loop behavior close to that designed for the approximate smooth plant. There are many examples in which any smooth approximation to nonsmooth nonlinear dynamics produces qualitatively different behavior.

As a matter of fact, a naïve application of the above approach for designing variable structure controllers, i.e., reduction to normal form, smooth ε-approximation of the nonsmooth friction, then variable structure control design (sliding and reaching control), will almost certainly fail. We will give a simple explanation below. As an alternative, we will use a backstepping approach, introduced in [12] for adaptive control design and adapted for recursive Lyapunov design in [13]. Now, let us consider the following simple example which highlights the essential issues.

Example 1: Sandwiched Friction

Suppose we reduce the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\phi_{fr}(x_2) + x_3 \\ \dot{x}_3 &= u\end{aligned}$$

to normal form. Let us write the friction model in the form of a nominal plus an uncertain part: $\phi_{fr}(x_2) = \phi_{fr0}(x_2) + \delta\phi_{fr}(x_2)$, where $\phi_{fr0}(x_2)$ is smooth. For example, $\phi_{fr0}(x_2) = \tanh(x_2 / \varepsilon), \varepsilon > 0$ and $\delta\phi_{fr}(x_2) = \mathrm{sign}(x_2) - \tanh(x_2 / \varepsilon)$.

Then we have the coordinate transform

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= -\phi_{fr}(x_2) + x_3\end{aligned} \qquad \text{which yields the transformed system} \qquad \begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -\phi'_{fr0}(z_2) + \delta\phi'_{fr}(z_2) + u\end{aligned}$$

Thus, any error in the friction function produces an uncertainty that depends on the derivative $\delta\phi'_{fr}(z_2)$. Obviously, if the friction function is nondifferentiable, this will produce an unbounded (although matched) uncertainty. The variable structure control, which has bounded control authority, cannot be made robust to this type of unbounded uncertainty. See Figure 3 for simulation results.

Let us instead base the normal form reduction on the smooth nominal system. Then we have the coordinate transform

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= -\phi_{fr0}(x_2) + x_3\end{aligned} \qquad \text{which yields the transformed system} \qquad \begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \delta\phi_{fr}(z_2) \\ \dot{z}_3 &= -\phi'_{fr0}(z_2) + u\end{aligned}$$

Now we have a bounded, although not matched, uncertainty. It is precisely because the uncertainty is unmatched that we use a backstepping approach. Before proceeding with this example we describe the backstepping process.

4 The VS Backstep Procedure

We give a brief description of the backstepping procedure we propose for SISO VS control system design in the presence of uncertain nonsmooth nonlinearities. The key innovations in our approach for nonsmooth plants are (1) that the states are grouped depending on where an uncertainty enters the system and the robustification is attempted only where the uncertainty is identified, and (2) that the control designed at each step is a variable structure control.

Consider a SISO nonlinear system in the (multi-state back-stepping) form:

$$\begin{aligned} x_i^{(n_i)} &= x_{i+1} + \Delta_i(x,t), \quad i = 1,\ldots,p-1 \\ x_p^{(n_p)} &= \alpha(x) + \rho(x)u + \Delta_p(x,t) \\ y &= x_1 \end{aligned} \tag{6}$$

We assume that the (possibly nonsmooth) uncertainties $\Delta_i(x,t)$ are bounded by smooth, non-negative functions $\varepsilon_i(x)$, i.e.,

$$0 \le \left|\Delta_i(x,t)\right| \le \varepsilon_i(x), \quad \forall t \tag{7}$$

Such a model might arise by reduction of a smooth nominal system to regular from and applying the transformation to the uncertain system.

At each of p-1 stages we design a 'pseudo-control' v_i and at the last (p) stage we design the actual control. The k^{th} control is obtained by designing a stabilizing smoothed VS controller for a ('nominal') system in the form:

$k = 1$

$$\begin{aligned} x_1^{(n_1)} &= v_1 \\ y_1 &= x_1 \end{aligned} \tag{8a}$$

$k = 2$

$$\begin{aligned} y_1^{(n_1)} &= x_2 \\ x_2^{(n_2)} &= v_2 \\ y_2 &= x_2 - v_1 \end{aligned} \tag{8b}$$

$k = 3, \ldots p-1$

$$\begin{aligned} y_i^{(n_i)} &= y_{i+1}, \quad i = 1, \ldots, k-2 \\ y_{k-1}^{(n_{k-1})} &= x_k, \\ x_k^{(n_k)} &= v_k \\ y_k &= x_k - v_{k-1} \end{aligned} \tag{8c}$$

$k = p$

$$\begin{aligned} y_i^{(n_i)} &= y_{i+1}, \quad i = 1, \ldots, p-2 \\ y_{p-1}^{(n_{p-1})} &= x_p, \\ x_p^{(n_p)} &= \alpha + \rho v_p \\ y_p &= x_p - v_{p-1} \end{aligned} \tag{8d}$$

To design the control v_k we first reduce the system (8) to normal form by successive differentiation:

$$y_k^{(n_k)} = v_k - L_f^{n_k}(x_k - v_{k-1}) \tag{9}$$

Thus, we identify the evolution equation in the new coordinate y_k that will replace x_k. Notice that the zero dynamics of this system are

$$\begin{aligned} y_i^{(n_i)} &= y_{i+1}, \quad i = 1, \ldots, k-2 \\ y_{k-1}^{(n_{k-1})} &= v_{k-1} \end{aligned} \tag{10}$$

Now, we design a VS stabilizing controller, $v_k(y_k, \ldots, y_k^{(n_k)})$ such that $y_k(t) \to 0$ as $t \to \infty$. For each $k < p$ we smooth the controller so that the process can be continued. Working in this way through the p stages, and redefining the states ($x \to y$) at each stage we arrive at the final set of dynamical equations.

$$\begin{aligned} y_i^{(n_i)} &= y_{i+1} + v_i(y_i, \ldots y_i^{(n_i)}) \quad i = 1, \ldots, p-1 \\ y_p^{(n_p)} &= \alpha + \rho u(y_p, \ldots y_p^{(n_p)}) \end{aligned} \tag{11}$$

Finally, we

Notice the triangular structure. This structure is illustrated in Figure 1. The same set of transformations applied to the set of equations for the actual system yield

$$\begin{aligned} y_i^{(n_i)} &= y_{i+1} + \Delta_i + v_i(y_i, \ldots y_i^{(n_i)}) \quad i = 1, \ldots, p-1 \\ y_p^{(n_p)} &= \alpha + \Delta_p + \rho u(y_p, \ldots y_p^{(n_p)}) \end{aligned} \tag{12}$$

The idea for establishing stability is roughly as follows. A VS controller is designed for system *p*, (11), via methods described above. The system is stable if and only if the zero dynamics,

$$y_i^{(n_i)} = y_{i+1} + v_i(y_i, \ldots y_i^{(n_i)}) \quad i = 1, \ldots, p-1, \tag{13}$$

are stable. But, v_{p-1} is itself a (smoothed) VS control so that (13) is stable if its zero dynamics:

$$y_i^{(n_i)} = y_{i+1} + v_i(y_i, \ldots y_i^{(n_i)}) \quad i = 1, \ldots, p-2, \tag{14}$$

are stable. The argument proceeds in this way.

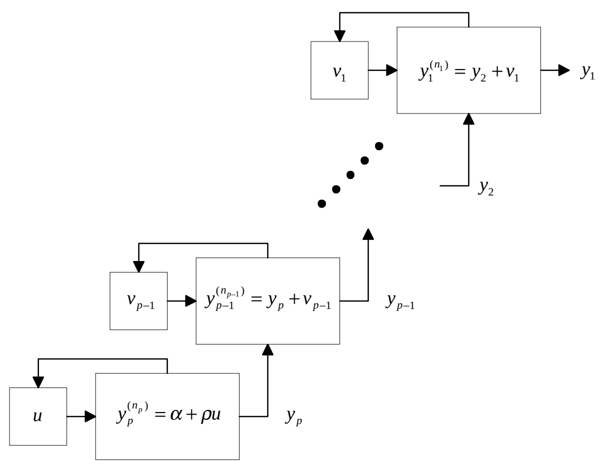

Figure 1. The triangular structure of the closed loop dynamics achieved with the multistate backstep control design.

Proposition 2: Consider the system (6) and suppose the uncertainties Δ_i satisfy the inequality (7) with continuous bounding functions ε_i, and α also has a continuous bounding function σ_α. Suppose that a controller is designed via the backstepping procedure of (8-11) and each control v_k, $k = 1, \ldots, p$, is a smoothed variable structure controller designed in accordance with the assumptions of Proposition 1.

Then for any given $\delta > 0$ there is a sufficiently small smoothing parameter $\varepsilon > 0$ such that all trajectories enter the ball $\|y\| < \delta$.

Proof: The *p*-th system

$$y_p^{(n_p)} = \alpha + \Delta_p + \rho v_p(y_p, \ldots y_p^{(n_p)}) \tag{15}$$

satisfies the conditions of Proposition 1 with $z_i = y_p^{(i-1)}$, $i = 1, \ldots, n_p$. Hence, we conclude that y_p (and its $n_p - 1$ derivatives) will be driven, in finite time, into a δ-neighborhood of the origin with a suitably small smoothing parameter. Now, the *p*-1 system is

$$y_{p-1}^{(n_{p-1})} = y_p(t) + \Delta_{p-1} + v_{p-1}(y_{p-1}, \ldots y_{p-1}^{(n_{p-1})}) \tag{16}$$

and $|y_p(t)| \le \delta$, $\forall t > t' < \infty$. Thus, we can incorporate $y_p(t)$ into $\Delta_{p-1}(x,t)$. It follows that (16) satisfies the conditions of Proposition 1 for $t > t'$, $z_i = y_{p-1}^{(i-1)}$, $i = 1, \ldots, n_{p-1}$, so that y_{p-1} (and its $n_{p-1} - 1$ derivatives) will be driven, in finite time, into a δ-neighborhood of the origin with a suitably small smoothing parameter. We continue in this way for systems $k = p-2, \ldots, 1$ to establish the conclusion of the theorem. ■

To implement the design process, we need to be able to reduce the system to normal form, compute an appropriate switching surface, assemble the switching control and insert smoothing and/or moderating functions as desired. Symbolic computing tools have been developed for this purpose [14]. We have included a function `SlidingSurface` that implements two alternatives for generating the sliding surface depending on the arguments provided. The function may be called via

```
{rho,s}=SlidingSurface[f,g,h,x,lam]
```

or

```
s=SlidingSurface[rho,vro,z,lam]
```

In the first case the data provided is the nonlinear system definition *f*, *g*, *h*, *x* and an *m*-vector lam which contains a list of desired exponential decay rates, one for each channel. The function returns the decoupling matrix rho and the switching surfaces *s* as functions of the state *x*. The matrix *K* is obtained by solving the appropriate Ricatti equation.

The second use of the function assumes that the input-output linearization has already been performed so that the decoupling matrix rho, the vector relative degree and the normal coordinate (partial) transformation $z(x)$ are known. In this case the dimension of each of the m switching surfaces is known so that it is possible to specify a complete set of eigenvalues for each surface. Thus, lam is a list of m-sublists containing the specified eigenvalues. Only the switching surfaces are returned. In this case K is obtained via pole placement.

The function `SwitchingControl[rho,s,bounds,Q,opts]` returns the variable structure control, where rho is the decoupling matrix, s is the vector of switching surfaces, 'bounds' is a list of controller bounds each in the form {lower bound, upper bound}, Q is an mxm positive definite matrix (a design parameter), and 'opts' are options that allow the inclusion of smoothing and/or moderating functions in the control.

Smoothing functions are specified by a rule of the form

```
SmoothingFunctions[x_]->{function1[x],...,functionm[x]}
```

Where m is the number of controls. The smoothing function option replaces any pure switch sign by a smooth switch function as specified. The following example makes use of these computatons.

Example 1 continued

Since the example system is already in multi-state back stepping form (6) no transformation is necessary. We break the system into two parts, treating x_3 as a temporary control and ignoring the uncertainty:

Step 1 Design a *smoothed* VS control, $v(x_1, x_2)$, for:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\phi_{fr}(x_2) + v \\ y &= x_1 \end{aligned}$$

Then, we design a VS control for the composite nominal system with modified output equation.

Step 2 Design a VS control, u for

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\phi_{fr}(x_2) + x_3 \\ \dot{x}_3 &= u \\ y &= x_3 - v(x_1, x_2) \end{aligned}$$

The Mathematica code is shown below where $[x_1, x_2, x_3] \rightarrow [\text{theta}, \text{omega}, \text{uu}]$. Using the previously described tools we have for Step 1:

```
f1 = {omega, -Tanh[omega / .02]};
g1 = {0, 1};
h1 = {theta};

{rho1, s1} = SlidingSurface[f1, g1, h1, {theta, omega}, {2}];

ctrlbnds = {{-5, 5}};
Q = {{1}};
vsc0 = SwitchingControl[rho1, s1, ctrlbnds, Q,
  SmoothingFunctions[x_] -> {Tanh[x / .01]}]

{-5 Tanh[803.066 omega + 1618.44 theta]}
```

and Step 2:

```
f = {omega, -Tanh[omega / .02] + uu , 0};
g = {0, 0, 1};
h = {uu - vsc0[[1]]};

{rho2, s2} = SlidingSurface[f, g, h, {theta, omega, uu}, {2}];

ctrlbnds = {{-5, 5}};
Q = {{1}};

vsc1 = SwitchingControl[rho2, s2, ctrlbnds, Q,
  SmoothingFunctions[x_] -> {Tanh[x / .01]}]

{-5 Tanh[423.607 (uu + 5 Tanh[803.066 omega + 1618.44 theta])]}
```

Simulation results obtained with this controller are illustrated by the trajectory in Figure 2. For comparison purposes, Figure 3 illustrates the failure of the non-backstepping controller to eliminate the position output error – as anticipated.

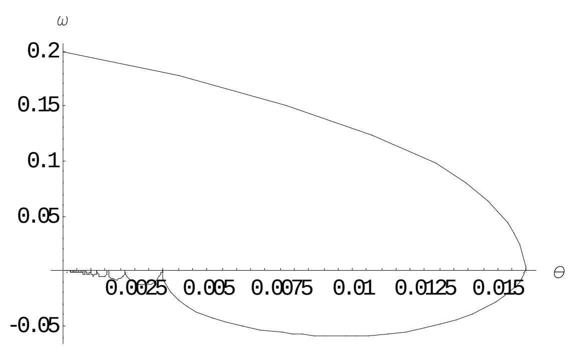

Figure 2. The projection of a state trajectory on the $\omega - \theta$ plane illustrates asymptotic convergence.

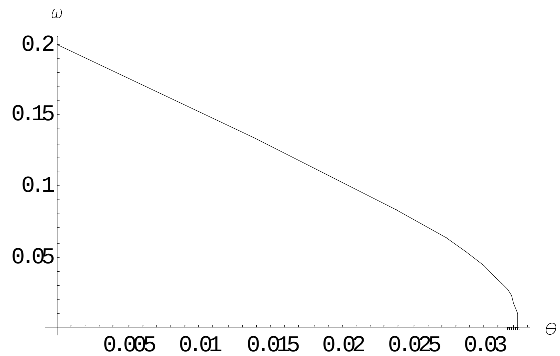

Figure 3. A similar projection using the conventional (non-backstep) design illustrates how the trajectory "sticks" because of the large matched uncertainty.

5 Conclusions

In this paper we have introduced a new method for design of control systems for a class of SISO systems with nondifferentiable, uncertain nonlinearities such as friction. The resulting controls are smoothed, variable structure controllers designed using a multi-state backstepping procedure. In preliminary studies the controller appears to be effective in dealing with the difficult problem of friction sandwiched between dynamical elements. Very little needs to be known about the details of the friction model. Only bounds on the friction function are required.

Ongoing work includes simulation and experimental studies of precision pointing problems in which friction significantly degrades performance, including comparisons with alternative approaches to friction compensation.

6 References

1. Friedland, B., *Advanced Control System Design*. 1996, Englewood Cliffs: Prentice hall.
2. Tao, G. and P.V. Kokotovic, *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*. 1996, New York: John Wiley and Sons, Inc.
3. Bennett, W.H., *et al.*, *Nonlinear and Adaptive control of Flexible Space Structures.* Transactions ASME, Journal of Dynamic Systems, Measurement and Control, 1993. **115**(1): p. 86–94.

4. Bennett, W.H., H.G. Kwatny, and M.J. Baek, *Nonlinear Dynamics and Control of Articulated Flexible Spacecraft: Application to SSF/MRMS.* AIAA Journal on Guidance, Control and Dynamics, 1994. **17**(1): p. 38–47.
5. Blankenship, G.L., *et al.*, *Integrated tools for Modeling and Design of Controlled Nonlinear Systems.* IEEE Control Systems, 1995. **15**(2): p. 65-79.
6. Kwatny, H.G. and H. Kim, *Variable Structure Regulation of Partially Linearizable Dynamics.* Systems & Control Letters, 1990. **15**: p. 67–80.
7. Kwatny, H.G. and J. Berg, *Variable Structure Regulation of Power Plant Drum Level*, in *Systems and Control Theory for Power Systems*, J. Chow, R.J. Thomas, and P.V. Kokotovic, Editors. 1995, Springer–Verlag: New York. p. 205-234.
8. Luk'yanov, A.G. and V.I. Utkin, *Methods of Reducing Equations of Dynamic Systems to Regular Form.* Avtomatica i Telemechanika, 1981(4): p. 5–13.
9. Marino, R., *High Gain Feedback Non–Linear Control Systems.* International Joural of Control, 1985. **42**(6): p. 1369–1385.
10. Kwatny, H.G., *Variable Structure Control of AC Drives*, in *Variable Structure Control for Robotics and Aerospace Applications*, K.D. Young, Editor. 1993, Elsevier: Amsterdam.
11. Utkin, V.I., *Sliding Modes and Their Application*. 1974 (in Russian) 1978 (in English), Moscow: MIR.
12. Kanellakapoulos, I., P.V. Kokotovic, and A.S. Morse, *Systematic design of Adaptive Controllers for Feedback Linearizable Systems.* IEEE Transactions on Automatic Control, 1991. **AC–36**(11): p. 1241–1253.
13. Freeman, R.A. and P.V. Kokotovic, *Design of 'Softer' Robust Nonlinear Control Laws.* Automatica, 1993. **29**(6): p. 1425-1437.
14. Teolis, C., H.G. Kwatny, and M. Mattice. *Symbolic Tools for Variable Structure Control System Design*. in *5th International Workshop on Variable Structure Systems*. 1998. Longboat Key, FL.